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Spacetime scale-invariance and the super p-brane

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Abstract. We generalize to p-dimensional extended objects and type II superstrings a recently proposed Green–Schwarz type I superstring action in which the tension T emerges as an integration constant of the equations of motion. The action is spacetime scale-invariant but its equations of motion are equivalent to those of the standard super p-brane for $T \neq 0$ and the null super p-brane for $T = 0$. We also show that for $p = 1$ the action can be written in 'Born–Infeld' form.

1. Introduction

The action for a particle of mass m in d-dimensional Minkowski spacetime with coordinates $\{x^m, m = 0, 1, \ldots, d-1\}$ is

$$S = \int \mathrm{d}t \left[\frac{1}{2e}\dot{x}^m \dot{x}^n \eta_{mn} - m^2 e\right] \tag{1}$$

where $e(t)$ is the worldline einbein and η_{mn} the (mostly plus) Minkowski metric. This action is invariant under Poincaré transformations in the d-dimensional target space but not under scale (or conformal-boost) transformations. However, this lack of scale invariance may be viewed, from the point of view of a *massless* particle in a $(d+1)$-dimensional spacetime, as a consequence of a particular choice of solution of the equations of motion. To see this, suppose that y is the coordinate of the extra dimension and write the action as

$$S = \int \mathrm{d}t \frac{1}{2e}[\dot{x}^m \dot{x}^n \eta_{mn} + \dot{y}^2]. \tag{2}$$

The y equation of motion is $\partial_t(e^{-1}\dot{y}) = 0$, i.e. $\dot{y} = me$ for arbitrary mass parameter m. The remaining equations are then the same as those of (1). This illustrates the fact that a massive particle can be viewed as a massless one in a higher dimension, with the mass interpreted as the component of momentum in the extra dimension. In the quantum theory the mass m is quantized if y is periodic and a choice of m then amounts to a truncation of a Kaluza–Klein theory. The variable y can in this case be viewed as parametrizing the fibre of a $U(1)$ bundle over (d-dimensional) spacetime.

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The Nambu–Goto action for a string, or more generally a p-brane, is analogous to that of the *massive* particle. To bring out this analogy it is convenient to write the p-brane action in the form

$$S = \int \mathrm{d}^{p+1}\xi \left\{ \frac{1}{2V} \det(\partial_i x^m \partial_j x^n \eta_{mn}) - T^2 V \right\} \tag{3}$$

where $\{\xi^i, i = 0, 1, \ldots, p\}$ are the worldvolume coordinates, $V(\xi)$ is an independent worldvolume density, and T is the tension (with units of mass/unit p-volume). As for the massive particle this action is also *not* scale invariant. It is natural to wonder what the analogue of (2) is in this case. This question was addressed in two recent papers [1, 2]. In [1] an additional variable, analogous to $y(t)$, was introduced, with the interpretation as the coordinate of the fibre of a $U(1)$ bundle over loop superspace [3] (or its extension to the space of maps of a p-brane to superspace). In this formulation the tension appears as an integration constant of the $y(t)$ equation of motion and can be interpreted as the momentum along the $U(1)$ fibre. However, the action proposed in [1] is not local on the worldsheet/worldvolume. It was shown subsequently [2] for $p = 1$ that the appropriate local generalization of (2) is an action containing an independent worldsheet 'electromagnetic' gauge field. We may readily generalize this to a p-brane action containing an independent p-form gauge potential

$$A = \frac{1}{p!} \mathrm{d}\xi^{i_p} \ldots \mathrm{d}\xi^{i_1} A_{i_1 \ldots i_p} \tag{4}$$

where the wedge product of differential forms is understood. Its $(p+1)$-form field-strength is†

$$F = \mathrm{d}A = \frac{1}{(p+1)!} \mathrm{d}\xi^{i_{p+1}} \ldots \mathrm{d}\xi^{i_1} F_{i_1 \ldots i_{p+1}} \tag{5}$$

and the corresponding action is

$$S = \int \mathrm{d}^{p+1}\xi \; \frac{1}{2V} [\det(\partial_i x \cdot \partial_j x) + 4\tilde{F}^2] \tag{6}$$

where $\tilde{F} = (1/(p+1)!)\epsilon^{i_{p+1} \ldots i_1} F_{i_1 \ldots i_{p+1}}$. The equation of motion for $A_{i_1 \ldots i_p}$ is $\partial_i (V^{-1} \tilde{F}) = 0$. Choosing the solution $\tilde{F} = \frac{1}{2} TV$ one then finds that the remaining field equations are those of (3). Moreover, the new action (6) has the *target space scale invariance*‡

$$x^m \to \lambda x^m \qquad A_{i_1 \ldots i_p} \to \lambda^{p+1} A_{i_1 \ldots i_p} \qquad V \to \lambda^{2(p+1)} V \tag{7}$$

which is broken by the solution $\tilde{F} = \frac{1}{2} TV$ if $T \neq 0$. This is entirely analogous to the particle case. In fact, for $p = 0$ one has $\tilde{F} = \frac{1}{2} \dot{A}$ and we recover (2) on identifying $e = V$ and $y = A$. Note that if the $\tilde{F}^2$ term in (6) is omitted we have

† Note that $F_{i_1 \ldots i_{p+1}} = (p+1)\partial_{[i_1} A_{i_2 \ldots i_{p+1}]}$ since we adopt the conventions that, for p-form P and q-form Q, $\mathrm{d}(PQ) = P\mathrm{d}Q + (-)^q (\mathrm{d}P)Q$.

‡ This should not be confused with the *worldvolume* scale invariance of certain formulations of the (super) p-brane [5].

the action of the *null* p-brane [4]. The action (6) can, therefore, be viewed as a kind of 'higher-dimensional' extension of the null p-brane, just as for $p = 0$ it is a higher-dimensional massless particle.

Consider now the supersymmetric extension of these ideas. For example, the action for the $d = 9$ massive superparticle is

$$S = \int \mathrm{d}t \left[\frac{1}{2e}\omega \cdot \omega - m^2 e + m\bar{\theta}\dot{\theta}\right] \tag{8}$$

where $\omega^m = \dot{x}^m - \mathrm{i}\bar{\theta}\Gamma^m\dot{\theta}$ and $\bar{\theta} = \theta^T C$ where C is the *symmetric* charge conjugation matrix. Note that the last term in (8) is *not* manifestly supersymmetric and can be interpreted as a Wess–Zumino term. As a consequence of this term, the mass m appears as a central charge in the supersymmetry algebra [6]. Central charges have a natural interpretation as components of momentum in 'extra' dimensions, and this suggests that it should be possible to derive the action of the nine-dimensional massive superparticle from the action of the massless superparticle in ten dimensions. The latter can be written in the form

$$S = \int \mathrm{d}t \frac{1}{2e}[\omega \cdot \omega + (\omega^9)^2] \tag{9}$$

where $\omega^9 = \dot{y} - \mathrm{i}\bar{\theta}\Gamma^9\dot{\theta}$. This is the supersymmetric extension of (2). The y equation of motion is $\partial_t(e^{-1}\omega^9) = 0$ which has the solution $\omega^9 = me$. As for the bosonic case the remaining equations of motion are those of (8), but note that the ten-dimensional action is *manifestly* supersymmetric, as there is no Wess–Zumino term, and has no dimensionful parameters.

For $p > 0$ there is a similar Wess-Zumino term in the standard super p-brane action and this leads to the appearance of a p-form topological charge in the supersymmetry algebra which is non-zero for spacetimes with non-trivial p-cycles [7]. This topological charge is obviously not central with respect to the d-dimensional Poincaré group, but is central with respect to the *global* symmetry group of spacetime which, in such cases, is always a proper subgroup of the d-dimensional Poincaré group. These topological charges again suggest the existence of some kind of 'higher-dimensional' manifestly supersymmetric action, without dimensionful parameters. Such an action was given in [1] but it contains variables that are not defined locally on the worldvolume. From the above discussion of the bosonic case one can guess that a local action with the required properties may be found by supersymmetrization of (6). This turns out to be the case. The resulting action is

$$S = \int \mathrm{d}^{p+1}\xi \frac{1}{2V}(g + \Phi^2) \tag{10}$$

where $g = \det(\Pi_i \cdot \Pi_j)$ with $\Pi_i^m = \partial_i x^m - \mathrm{i}\bar{\theta}\Gamma^m\partial_i\theta$ and Φ is the dual of a supertranslation-invariant 'modified' field strength for a worldvolume p-form gauge potential A. As a result of this modification A acquires a non-trivial supersymmetry transformation.

An action of the form (10) was given in [2] for the $N = 1$ superstring, where it was derived from a free-differential algebra extension of the supertranslation algebra. In this paper we consider the general p case and type II superstrings, and we discuss some features of this formulation not mentioned previously, such as scale invariance.

We hope to persuade the reader that the new scale-invariant formulation of the super p-brane action presented here is a natural one. This is especially true for $p = 1$ because the action (10) may in this case be cast in a geometrically suggestive 'Born–Infeld' form, as we shall show in section 4. For the $p = 0$ case there is the additional bonus that the massless particle can be quantized covariantly using twistor methods [8]. One of the motivations for the work reported here is the hope that, given a scale-invariant super p-brane action, twistor methods might again be applicable. In fact, progress along these lines has recently been announced [9].

2. The free differential superalgebra

We shall begin, as in [2], with the Maurer–Cartan equations,

$$\mathrm{d}\psi = 0 \qquad \mathrm{d}\Pi^m - \mathrm{i}\bar{\psi}\Gamma^m\psi = 0 \tag{11}$$

for the d-vector 1-form Π^m and Grassmann-odd spinor 1-form ψ of the supertranslation algebra. The exterior product of forms is again understood in (11) and in what follows. We now extend this algebra to a free differential superalgebra [10] by the introduction of an additional $(p+1)$-form F subject to

$$\mathrm{d}F + h(\Pi, \psi) = 0 \tag{12}$$

where h is a closed $(p+2)$-form constructed from Π and ψ. We choose

$$h = \frac{\mathrm{i}}{2p!}\Pi^{m_p}\ldots\Pi^{m_1}\bar{\psi}\Gamma_{m_1\ldots m_p}\psi \tag{13}$$

which is closed for the values of (p, d) admitted by the 'branescan' [11]. For simplicity, we shall assume that ψ is a Majorana spinor and hence restrict ourselves to $p = 1, 2$ and 5, but this covers most of the interesting cases.

Observe that h cannot be written as $h = \mathrm{d}b$ if we require that b be constructed from Π^m and ψ. This means that h represents a *non-trivial* class of the $(p+2)$th equivariant cohomology group of the supertranslation algebra [12]. As a consequence it is not possible to set $F = F' + \mathrm{d}K$ in such a way that (12) reduces to $\mathrm{d}F' = 0$. The free differential superalgebra defined by (11) and (12) is therefore a non-trivial extension of the supertranslation algebra.

Equations (11) and (12) may be solved as follows in terms of the 0-forms $Z^M = (x^m, \theta^\alpha)$ and a p-form A, which may be viewed as the coordinates of the 'group manifold' $\tilde{\Sigma}$ associated with the free differential superalgebra and extending the supertranslation group manifold Σ:

$$\psi = \mathrm{d}\theta \qquad \Pi^m = \mathrm{d}x^m - \mathrm{i}\bar{\theta}\Gamma^m\mathrm{d}\theta \qquad F = \mathrm{d}A - b. \tag{14}$$

Here b is a potential for h, i.e. $h = \mathrm{d}b$. From earlier remarks it should be clear that b cannot be written entirely in terms of ψ and Π^m but must involve x^m and/or θ explicitly. In fact, one can always arrange for x^m to appear as $\mathrm{d}x^m$ at the cost of undifferentiated θs. For such a choice b will be translation, but not supersymmetry, invariant. This lack of supersymmetry invariance of b is restricted by the fact that $\mathrm{d}b$ *is* invariant, from which it follows that $\delta_\epsilon b = \mathrm{d}(\mathrm{i}\bar{\epsilon}\Delta)$ for some p-form Δ. The (modified) field strength F will then be invariant if we ascribe to A the supersymmetry variation

$$\delta_\epsilon A = \mathrm{i}\bar{\epsilon}\Delta . \tag{15}$$

To find Δ we observe that, for an arbitrary variation δZ^M,

$$\begin{aligned}\delta b &= \mathrm{d}\left(\frac{1}{p!}\Pi^{M_{p+1}}\dots\Pi^{M_2}(\delta Z^{M_1})b_{M_1\dots M_{p+1}}\right)\\ &\qquad + \frac{1}{(p+1)!}\Pi^{M_{p+1}}\dots\Pi^{M_1}\delta Z^N h_{NM_1\dots M_{p+1}}\\ &= \mathrm{d}\left(\frac{1}{p!}\Pi^{A_p}\dots\Pi^{A_1}(\delta Z)^B b_{BA_1\dots A_p}\right)\\ &\qquad + \frac{1}{(p+1)!}\Pi^{A_{p+1}}\dots\Pi^{A_1}(\delta Z)^B h_{BA_1\dots A_{p+1}}\end{aligned} \tag{16}$$

where $(\delta Z)^A = ((\delta x^m + \mathrm{i}\delta\bar{\theta}\Gamma^m\theta)\delta_m^a, \delta\theta^\alpha)$. For flat superspace the only non-vanishing component of h is

$$h_{\alpha\beta a_1\dots a_p} = \mathrm{i}(\Gamma_{a_1\dots a_p})_{\alpha\beta} \tag{17}$$

and for a supersymmetry variation $(\delta_\epsilon Z)^A = (-2\mathrm{i}\bar{\theta}\Gamma^a\epsilon, \epsilon^\alpha)$. In this case (16) reduces to

$$\begin{aligned}\delta_\epsilon b &= \mathrm{d}\left(\frac{1}{p!}\Pi^{A_p}\dots\Pi^{A_1}(\delta_\epsilon Z)^B b_{BA_1\dots A_p}\right) + \frac{1}{p!}\mathrm{i}\Pi^{a_p}\dots\Pi^{a_1}(\mathrm{d}\bar{\theta}\Gamma_{a_1\dots a_p}\epsilon)\\ &\qquad + \frac{1}{(p-1)!}\Pi^{a_p}\dots\Pi^{a_2}(\mathrm{d}\bar{\theta}\Gamma_{a_1 a_2\dots a_p}d\theta)(\bar{\theta}\Gamma^{a_1}\epsilon)\\ &= \mathrm{d}\left[\frac{1}{p!}\Pi^{A_p}\dots\Pi^{A_1}(\delta_\epsilon Z)^B b_{BA_1\dots A_p} + \frac{1}{p!}\mathrm{i}\Pi^{a_p}\dots\Pi^{a_1}(\bar{\theta}\Gamma_{a_1\dots a_p}\epsilon)\right]\\ &\qquad + \frac{1}{(p-1)!}\Pi^{a_p}\dots\Pi^{a_2}[(\mathrm{d}\bar{\theta}\Gamma^{a_1}\mathrm{d}\theta)(\bar{\theta}\Gamma_{a_1\dots a_p}\epsilon)\\ &\qquad + (\mathrm{d}\bar{\theta}\Gamma_{a_1\dots a_p}\mathrm{d}\bar{\theta})(\bar{\theta}\Gamma^{a_1}\epsilon)]\end{aligned} \tag{18}$$

where $\mathrm{d}\Pi^a = \mathrm{i}\mathrm{d}\bar{\theta}\Gamma^a\mathrm{d}\theta$ has been used to arrive at the second equality. We now need to write the last term on the right-hand side of (18) as an exact form. The procedure for doing this makes repeated use of the identity

$$(\Gamma^{a_1})_{(\alpha\beta}(\Gamma_{a_1\dots a_p})_{\gamma\delta)} \tag{19}$$

which is equivalent to the closure of h. First, this identity implies that

$$\begin{aligned}&(\mathrm{d}\bar{\theta}\Gamma^{a_1}\mathrm{d}\theta)(\bar{\theta}\Gamma_{a_1\dots a_p}\epsilon) + (\mathrm{d}\bar{\theta}\Gamma_{a_1\dots a_p}\mathrm{d}\theta)(\bar{\theta}\Gamma^{a_1}\epsilon)\\ &\qquad = \tfrac{2}{3}\mathrm{d}[(\mathrm{d}\bar{\theta}\Gamma^{a_1}\theta)(\bar{\theta}\Gamma_{a_1\dots a_p}\epsilon) + (\mathrm{d}\bar{\theta}\Gamma_{a_1\dots a_p}\theta)(\bar{\theta}\Gamma^{a_1}\epsilon)].\end{aligned} \tag{20}$$

Using this in (18) we obtain

$$\delta_\epsilon b = \mathrm{d}\left\{\frac{1}{p!}\Pi^{A_p}\ldots\Pi^{A_1}(\delta_\epsilon Z)^B b_{BA_1\ldots A_p} + \frac{1}{p!}\mathrm{i}\Pi^{a_p}\ldots\Pi^{a_1}(\bar{\theta}\Gamma_{a_1\ldots a_p}\epsilon)\right.$$

$$\left. + \frac{2}{3(p-1)!}\Pi^{a_p\ldots a_2}\Big[(\mathrm{d}\bar{\theta}\Gamma^{a_1}\theta)(\bar{\theta}\Gamma_{a_1a_2\ldots a_p}\epsilon) + (\mathrm{d}\bar{\theta}\Gamma_{a_1a_2\ldots a_p}\theta)(\bar{\theta}\Gamma^{a_1}\epsilon)\Big]\right\}$$

$$+ \frac{2\mathrm{i}}{3(p-1)!}\Pi^{a_p}\ldots\Pi^{a_3}(\mathrm{d}\bar{\theta}\Gamma^{a_2}\mathrm{d}\theta)$$

$$\times\left[(\mathrm{d}\bar{\theta}\Gamma^{a_1}\theta)(\bar{\theta}\Gamma_{a_1a_2a_3\ldots a_p}\epsilon) + (\mathrm{d}\bar{\theta}\Gamma_{a_1a_2a_3\ldots a_p\epsilon}\theta)(\bar{\theta}\Gamma^{a_1}\epsilon)\right]. \qquad (21)$$

For $p = 1$ the last term is absent, so the 1-form Δ may be read from this expression (it agrees with the result of [2]). For $p > 1$ the procedure must be continued in order to rewrite this last term as an exact form. In practice it is simpler to write down the general form of $\delta_\epsilon b$ as an exact differential with arbitrary coefficients and then fix them by comparison with (16). The result for $p = 2$ may be found in [7]. Here we shall give the result for $p = 5$, starting from the following expression for the 6-form b [13]:

$$b = -\mathrm{i}(\bar{\theta}\Gamma_{\mu\nu\rho\sigma\lambda}d\theta)[\Pi^\mu\Pi^\nu\Pi^\rho\Pi^\sigma\Pi^\lambda + \mathrm{i}\tfrac{5}{2}\Pi^\mu\Pi^\nu\Pi^\rho\Pi^\sigma(\bar{\theta}\Gamma^\lambda\mathrm{d}\theta)$$

$$- \tfrac{10}{3}\Pi^\mu\Pi^\nu\Pi^\rho(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\theta}\Gamma^\lambda\mathrm{d}\theta) - \mathrm{i}\tfrac{5}{2}\Pi^\mu\Pi^\nu(\bar{\theta}\Gamma^\rho\mathrm{d}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\theta}\Gamma^\lambda\mathrm{d}\theta)$$

$$+ \Pi^\mu(\bar{\theta}\Gamma^\nu\mathrm{d}\theta)(\bar{\theta}\Gamma^\rho\mathrm{d}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\theta}\Gamma^\lambda\mathrm{d}\theta)$$

$$+ \mathrm{i}\tfrac{1}{6}(\bar{\theta}\Gamma^\mu\mathrm{d}\theta)(\bar{\theta}\Gamma^\nu\mathrm{d}\theta)(\bar{\theta}\Gamma^\rho\mathrm{d}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\theta}\Gamma^\lambda\mathrm{d}\theta)]. \qquad (22)$$

We find that $\delta_\epsilon b = \mathrm{d}(\bar{\epsilon}\Delta)$ where

$$\bar{\epsilon}\Delta = \mathrm{i}(\bar{\epsilon}\Gamma_{\mu\nu\rho\sigma\lambda}\theta)\Pi^\mu\Pi^\nu\Pi^\rho\Pi^\sigma\Pi^\lambda - \tfrac{25}{6}(\bar{\epsilon}\Gamma_{\mu\nu\rho\sigma\lambda}\theta)(\bar{\theta}\Gamma^\lambda\mathrm{d}\theta)\Pi^\mu\Pi^\nu\Pi^\rho\Pi^\sigma$$

$$+ \tfrac{5}{6}(\bar{\theta}\Gamma_{\mu\nu\rho\sigma\lambda}\mathrm{d}\theta)(\bar{\epsilon}\Gamma^\lambda\theta)\Pi^\mu\Pi^\nu\Pi^\rho\Pi^\sigma$$

$$- \mathrm{i}\tfrac{22}{3}(\bar{\epsilon}\Gamma_{\mu\nu\rho\sigma\lambda}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\theta}\Gamma^\lambda\mathrm{d}\theta)\Pi^\mu\Pi^\nu\Pi^\rho$$

$$- \mathrm{i}\tfrac{8}{3}(\bar{\theta}\Gamma_{\mu\nu\rho\sigma\lambda}\mathrm{d}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\epsilon}\Gamma^\lambda\theta)\Pi^\mu\Pi^\nu\Pi^\rho$$

$$+ \tfrac{93}{14}(\bar{\epsilon}\Gamma_{\mu\nu\rho\sigma\lambda}\theta)(\bar{\theta}\Gamma^\rho\mathrm{d}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\theta}\Gamma^\lambda\mathrm{d}\theta)\Pi^\mu\Pi^\nu$$

$$- \tfrac{47}{14}(\bar{\theta}\Gamma_{\mu\nu\rho\sigma\lambda}\mathrm{d}\theta)(\bar{\theta}\Gamma^\rho\mathrm{d}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\epsilon}\Gamma^\lambda\theta)\Pi^\mu\Pi^\nu$$

$$+ \mathrm{i}\tfrac{193}{63}(\bar{\epsilon}\Gamma_{\mu\nu\rho\sigma\lambda}\theta)(\bar{\theta}\Gamma^\nu\mathrm{d}\theta)(\bar{\theta}\Gamma^\rho\mathrm{d}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\theta}\Gamma^\lambda\mathrm{d}\theta)\Pi^\mu$$

$$+ \mathrm{i}\tfrac{122}{63}(\bar{\theta}\Gamma_{\mu\nu\rho\sigma\lambda}\mathrm{d}\theta)(\bar{\theta}\Gamma^\nu\mathrm{d}\theta)(\bar{\theta}\Gamma^\rho\mathrm{d}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\epsilon}\Gamma^\lambda\theta)\Pi^\mu$$

$$- \tfrac{793}{1386}(\bar{\epsilon}\Gamma_{\mu\nu\rho\sigma\lambda}\theta)(\bar{\theta}\Gamma^\mu\mathrm{d}\theta)(\bar{\theta}\Gamma^\nu\mathrm{d}\theta)(\bar{\theta}\Gamma^\rho\mathrm{d}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\theta}\Gamma^\lambda\mathrm{d}\theta)$$

$$+ \tfrac{593}{1386}(\bar{\theta}\Gamma_{\mu\nu\rho\sigma\lambda}\mathrm{d}\theta)(\bar{\theta}\Gamma^\mu\mathrm{d}\theta)(\bar{\theta}\Gamma^\nu\mathrm{d}\theta)(\bar{\theta}\Gamma^\rho\mathrm{d}\theta)(\bar{\theta}\Gamma^\sigma\mathrm{d}\theta)(\bar{\epsilon}\Gamma^\lambda\theta). \qquad (23)$$

3. The spacetime scale-invariant super p-brane

Let W be the worldvolume of a p-dimensional extended object. Given a map $f : W \to \bar{\Sigma}$, we can pull back the *supersymmetry invariant* forms (14) to the worldvolume:

$$f^*(\xi) = \mathrm{d}\xi^i \partial_i \theta \qquad f^*(\Pi^m) = \mathrm{d}\xi^i \Pi_i^m$$
$$f^*(F) = \frac{1}{(p+1)!} \mathrm{d}\xi^{i_{p+1}} \dots \mathrm{d}\xi^{i_1} F_{i_1 \dots i_{p+1}} \tag{24}$$

where $\Pi_i^m = \partial_i x^m - \mathrm{i}\bar{\theta}\Gamma^m \partial_i \theta$ and

$$F_{i_1 \dots i_{p+1}} = (p+1)\partial_{[i_1} A_{i_2 \dots i_{p+1}]} - b_{i_1 \dots i_{p+1}} \tag{25}$$

with $b_{i_1 \dots i_{p+1}}$ the components of the pull-back $f^*(b)$ of b. We may now construct a manifestly supersymetric worldvolume metric as

$$g_{ij} = \Pi_i^m \Pi_j^n \eta_{mn}. \tag{26}$$

In addition, the 'modified' field strength $F_{i_1 \dots i_{p+1}}$ has only one independent component, which may be written as the (gauge-invariant and supersymmetric) worldvolume scalar density

$$\Phi = \frac{2}{(p+1)!} \epsilon^{i_{p+1} \dots i_1} F_{i_1 \dots i_{p+1}}. \tag{27}$$

By introducing an independent density V we can now write down the *manifestly* supersymmetric action

$$S = \int \mathrm{d}^{p+1}\xi \frac{1}{2V}(g + \Phi^2) \tag{28}$$

where g is the determinant of g_{ij}.

As for the bosonic action (6), this action is invariant under the target space scale transformations of (7) with

$$\theta \to \lambda^{1/2}\theta. \tag{29}$$

We shall see in the following that the equations of motion of our new action are equivalent to either those of the standard super p-brane or those of the null super p-brane, depending on the choice of an integration constant in the A equation of motion.

To obtain the field equations we need the variation of $b_{i_1 \dots i_p}$ induced by a general variation δZ^M of Z^M. From (16) this is

$$\frac{1}{(p+1)!} \epsilon^{i_{p+1} \dots i_1} \delta b_{i_1 \dots i_{p+1}}$$
$$= \frac{1}{p!} \epsilon^{i_{p+1} \dots i_1} \partial_{i_1} [(\delta Z)^A b_{A i_2 \dots i_{p+1}}] + \frac{1}{(p+1)!} \epsilon^{i_{p+1} \dots i_1} (\delta Z)^B h_{B i_1 \dots i_{p+1}} \tag{30}$$

where $h_{Bi_1\ldots i_{p+1}} = \Pi_{i_{p+1}}^{A_{p+1}} \ldots \Pi_{i_1}^{A_1} h_{BA_1\ldots A_{p+1}}$. Using the specific form of h given in (17) we find that

$$\begin{aligned}
&\frac{1}{(p+1)!}\epsilon^{i_{p+1}\ldots i_1}\delta b_{i_1\ldots i_{p+1}} \\
&\qquad = \frac{1}{p!}\epsilon^{i_{p+1}\ldots i_1}\partial_{i_1}[(\delta Z)^A b_{Ai_2\ldots i_{p+1}}] - \frac{\mathrm{i}}{p!}\epsilon^{i_p\ldots i_1 j}\partial_j\bar\theta\Gamma_{i_1\ldots i_p}\delta\theta \\
&\qquad\quad + \frac{\mathrm{i}}{2(p-1)!}\epsilon^{i_{p-1}\ldots i_1 jk}\partial_j\bar\theta\Gamma_a\Gamma_{i_1\ldots i_{p-1}}\partial_k\theta(\delta Z)^a.
\end{aligned} \tag{31}$$

By defining the matrix

$$\Xi = \frac{1}{(p+1)!}\epsilon^{i_{p+1}\ldots i_1}\Gamma_{i_1\ldots i_{p+1}} \tag{32}$$

which satisfies

$$\Xi^2 = -g \tag{33}$$

and using the relation

$$\epsilon^{i_{p+1}\ldots i_{k+1}i_k\ldots i_1}\Gamma_{i_{k+1}\ldots i_{p+1}} = (p-k+1)!\Gamma^{i_k\ldots i_1}\Xi \tag{34}$$

we can simplify (31) to

$$\begin{aligned}
&\frac{1}{(p+1)!}\epsilon^{i_{p+1}\ldots i_1}\delta b_{i_1\ldots i_{p+1}} \\
&\qquad = \frac{1}{p!}\epsilon^{i_{p+1}\ldots i_1}\partial_{i_1}[(\delta Z)^A b_{Ai_2\ldots i_{p+1}}] + \mathrm{i}\partial_j\bar\theta\Gamma^j\Xi\delta\theta + \frac{\mathrm{i}}{2}\partial_i\bar\theta\Gamma_a\Gamma^{ij}\Xi\partial_j\theta(\delta Z)^a.
\end{aligned} \tag{35}$$

It is now straightforward to derive the variation of the action (28) under a general variation of Z^M, $A_{i_1\ldots i_p}$ and V:

$$\begin{aligned}
\delta S = \int \mathrm{d}^{p+1}\xi\Big\{ &-\frac{\delta V}{2V^2}(g+\Phi^2) + 2\mathrm{i}\partial_i\bar\theta\Gamma^i\left(\frac{g}{V} - (\Phi V^{-1})\Xi\right)\delta\theta \\
&- \frac{2}{p!}\epsilon^{i_{p+1}\ldots i_1}(\delta A_{i_2\ldots i_{p+1}} - (\delta Z)^A b_{Ai_2\ldots i_{p+1}})\partial_{i_1}(V^{-1}\Phi) \\
&- (\delta Z)_a\left[\partial_i\left(\frac{g}{V}\Pi^{ia}\right) + (\mathrm{i}\Phi V^{-1})\partial_i\bar\theta\Gamma^a\Gamma^{ij}\Xi\partial_j\theta\right]\Big\}.
\end{aligned} \tag{36}$$

From this result it can be seen that, like the usual super p-brane action, the action (3) is invariant under the fermionic gauge transformation

$$\delta_\kappa x^m = -\mathrm{i}\delta_\kappa\bar\theta\Gamma^m\theta \qquad \delta_\kappa A_{i_1\ldots i_p} = (\delta_\kappa Z)^A b_{Ai_2\to i_1\ldots i_{p+1}\to i_p} \tag{37}$$

$$\delta_\kappa\theta = [(\Phi V^{-1}) + V^{-1}\Xi]\kappa \qquad \delta_\kappa V = \frac{4\mathrm{i}}{p!}\epsilon^{i_{p+1}\ldots i_2 i_1}\partial_{i_1}\bar\theta\Gamma_{i_2\ldots i_{p+1}}\kappa \tag{38}$$

where $\kappa(\xi)$ is a worldvolume scalar but spacetime spinor parameter. For a transformation of the type (37) only the first two terms in (36) survive and using (38) and $\Xi^2 = -g$ these are easily seen to cancel.

The $A_{i_1...i_p}$ field equation is $\partial_i(V^{-1}\Phi) = 0$. Choosing the solution

$$\Phi = VT \tag{39}$$

with $T \neq 0$, the remaining equations reduce to $V = (1/T)\sqrt{-g}$ and

$$(1+\Gamma)\Gamma^i\partial_i\theta = 0 \qquad \partial_i(\sqrt{-g}\Pi^{ia}) - \mathrm{i}\sqrt{-g}\partial_i\bar{\theta}\Gamma^a\Gamma^{ij}\Gamma\partial_j\theta = 0 \tag{40}$$

where Γ is the matrix

$$\Gamma = \Sigma/\sqrt{-g} \tag{41}$$

with the property that $\Gamma^2 = 1$. Equations (40) are precisely those of the standard super p-brane action [14].

If, on the other hand, we choose $\tilde{F} = 0$, then the remaining equations reduce to

$$g = 0 \qquad \partial_i(V^{-1}\tilde{g}^{ij}\Pi_j^a) = 0 \qquad \tilde{g}^{ij}\Gamma_i\partial_j\theta = 0 \tag{42}$$

where $\tilde{g}^{ij}$ is the matrix of co-factors of g_{ij}. These are the equations of motion of the null super p-brane which has the action [15]

$$S = \int \mathrm{d}^{p+1}\xi\frac{1}{2V}g. \tag{43}$$

Its κ transformations are those of (37) and (38) with $\Phi = 0$. We remark that the null super p-brane action is κ-invariant for any spacetime dimension, which shows that κ-symmetry and spacetime supersymmetry imply worldvolume supersymmetry only if $T \neq 0$.

4. Type II superstrings

Among supersymmetric extended object actions, the $p = 1$ case is special because there is the possibility of *extended* (non-minimal) supersymmetry, i.e. the type II Green–Schwarz (GS) superstring†. The spacetime scale-invariant reformulation of the action follows the same pattern as the general p, but type I, case just considered. However, some of the details differ so we shall now consider this case separately. We shall also take the opportunity to show how $p = 1$ is special in another respect; both the new type I and type II superstring actions may be rewritten in 'Born–Infeld' form.

We start from the $N = 2$ superspace closed 3-form

$$h = \frac{\mathrm{i}}{2}\Pi^m[(\mathrm{d}\bar{\theta}_1\Gamma_m d\theta_1) - (\mathrm{d}\bar{\theta}_2\Gamma_m\mathrm{d}\theta_2)] \tag{44}$$

† It has recently been suggested [16] that type II 5-branes and type II membranes may also be possible, by allowing worldvolume fields of spin $> 1/2$, but no κ-invariant spacetime-Poincaré invariant action of this type has been constructed yet.

where

$$\Pi^m = dx^m - i\bar{\theta}_1\Gamma^m d\theta_1 - i\bar{\theta}_2\Gamma^m d\theta_2. \tag{45}$$

For $d = 10$ the minimal spinor is chiral so that two type II actions are possible according to whether the $d = 10$ chirality of the spinors θ_1 and θ_2 is opposite (type IIA) or the same (type IIB) but for the analysis to follow it will not be necessary to specify the type.

We now introduce the additional 2-form $F = dA - b$, as in (14), where b is a potential for h of (44); a translation-invariant choice is

$$b = -\frac{i}{2}dx^m(\bar{\theta}_1\Gamma_m d\theta_1 - \bar{\theta}_2\Gamma_m d\theta_2) + \frac{1}{2}(\bar{\theta}_1\Gamma^m d\theta_1)(\bar{\theta}_2\Gamma_m d\theta_2). \tag{46}$$

Following the analysis of section 3 one can show that under the $N = 2$ supersymmetry transformation

$$\delta\theta_1 = \epsilon_1 \qquad \delta\theta_2 = \epsilon_2 \qquad \delta x^m = (i\bar{\epsilon}_1\Gamma^m\theta_1 + i\bar{\epsilon}_2\Gamma^m\theta_2) \tag{47}$$

the 2-form b acquires the transformation $\delta_\epsilon b = id(\bar{\epsilon}_1\Delta_1 - \bar{\epsilon}_2\Delta_2)$ where

$$\Delta_r = -\tfrac{1}{2}(dx^m + \tfrac{1}{3}i\bar{\theta}_r\Gamma^m d\theta_r)\Gamma_m\theta_r \qquad r = 1,2. \tag{48}$$

It follows that the 'modified' field strength F will be supertranslation (and Lorentz) invariant provided that we assign to the independent 1-form potential A the supersymmetry transformation

$$\delta A = i\bar{\epsilon}_1\Delta_1 - i\bar{\epsilon}_2\Delta_2. \tag{49}$$

The forms $(\Pi^m, d\theta_1^\alpha, d\theta_2^\alpha, F)$ can again be viewed as the (left)-invariant differential forms associated with a free-differential algebra.

From these invariant forms we can now construct similar worldsheet forms with components

$$\begin{aligned}
&\Pi_i^m = \partial_i x^m - i\bar{\theta}_1\Gamma^m\partial_i\theta_1 - i\bar{\theta}_2\Gamma^m\partial_i\theta_2 \\
&(\Pi_i^\alpha)_1 = \partial_i\theta_1^\alpha \qquad (\Pi_i^\alpha)_2 = \partial_i\theta_2^\alpha \\
&F_{ij} = \partial_i A_j - \partial_j A_i + [\tfrac{1}{2}i\partial_i x^m(\bar{\theta}_1\Gamma_m\partial_j\theta_1 - \bar{\theta}_2\Gamma_m\partial_j\theta_2) \\
&\qquad - \tfrac{1}{2}(\bar{\theta}_1\Gamma^m\partial_i\theta_1)(\bar{\theta}_2\Gamma_m\partial_j\theta_2) - (i \leftrightarrow j)]
\end{aligned} \tag{50}$$

from which we construct the worldsheet metric $g_{ij} = \sum_{r=1}^{2}(\Pi_i^m)_r(\Pi_j^n)_r\eta_{mn}$ and hence the worldsheet densities $\sqrt{-\det g_{ij}}$ and $\Phi = \epsilon^{ij}F_{ij}$. The spacetime scale-invariant type II superstring action may now be written exactly as in (28). However, for $p = 1$, this is equivalent to the 'Born–Infeld-type' action

$$S = \int d^2\xi \frac{1}{2V}\det(g_{ij} + 2F_{ij}) \tag{51}$$

since the cross terms between g_{ij} and F_{ij} in the expansion of the determinant cancel. Following the steps of section 3 one can show that this action has the κ-gauge invariance

$$\begin{aligned}
\delta_\kappa x^m &= -\mathrm{i}\delta_\kappa\bar\theta_1\Gamma^m\theta_1 - \mathrm{i}\delta_\kappa\bar\theta_2\Gamma^m\theta_2 \\
\delta_\kappa A_i &= \mathrm{i}\Pi_i^m(\bar\theta_1\Gamma_m\delta_\kappa\theta_1 - \bar\theta_2\Gamma_m\delta_\kappa\theta_2) + (\bar\theta_1\Gamma^m\partial_j\theta_1)(\bar\theta_2\Gamma_m\delta_\kappa\theta_2) \\
&\qquad - (\bar\theta_2\Gamma^m\partial_j\theta_2)(\bar\theta_1\Gamma_m\delta_\kappa\theta_1) \\
\delta_\kappa\theta_1 &= (g + \epsilon^{ij}F_{ij}\Xi)\kappa_1 \qquad \delta_\kappa\theta_2 = (g - \epsilon^{ij}F_{ij}\Xi)\kappa_2 \\
\delta_\kappa V &= 4\mathrm{i}Vg[(\partial_i\bar\theta_1\Gamma^i\kappa_1) + (\partial_i\bar\theta_2\Gamma^i\kappa_2)].
\end{aligned} \tag{52}$$

The type I action in this form is obtained simply by setting $\theta_2 = 0$.

5. Comments

We have emphasized that the new formulation of Green–Schwarz type actions is spacetime *scale* invariant. The massless particle is invariant under the full higher-dimensional conformal group (including conformal boosts). It is unclear whether there is an analogue of this larger group for $p \geqslant 1$.

We have concentrated in this paper on flat superspace but the results are readily generalized to curved space. Indeed, the action remains that of (10) but now

$$\Pi^A = \mathrm{d}Z^M E_M{}^A \tag{53}$$

where $E_M{}^A$ is the superspace supervielbein, and

$$F = \mathrm{d}A - B \tag{54}$$

where $H = \mathrm{d}B$ is the supergravity $(p+2)$-form. We expect that, as usual, κ-symmetry will require that the background supergravity fields satisfy their equations of motion. An interesting further question is whether this can be generalized in a κ-invariant way to include interactions with background Yang–Mills fields along the lines of [17].

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